Commuting Conjugacy Class Graphs of Symmetric Groups

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I wrote the preliminary version of this paper as an undergraduate. OpenAI Codex contributed new proofs and edits. See the *AI disclosure on page 7*.

Abstract

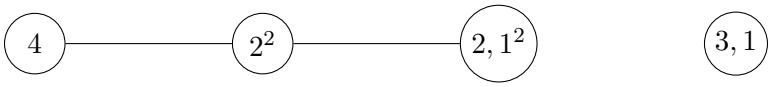
We study the graph on the nonidentity conjugacy classes of the symmetric group S_n, with two distinct classes joined when they contain commuting representatives. For $n \geq 3$ the graph is perfect exactly when $n \leq 7$ or $n = 9$. An induced pentagon in degree eight gives one in every degree $n \geq 13$ by adjoining a single cycle of length $n - 8$. The component containing the transpositions has diameter four for every $n \geq 8$. We determine when this is the whole graph. The logarithms of the clique and chromatic numbers both have order $\sqrt{n}$. The two numbers agree for $3 \leq n \leq 28$. Whether they agree in every degree remains open.

1 Introduction

Let G be a finite group. The *conjugacy class* of $g \in G$ is $g^G = \{xgx^{-1} : x \in G\}$. The *centre* $Z(G)$ consists of the elements commuting with every element of G. The vertices of the *commuting conjugacy class graph* $\Gamma(G)$ are the conjugacy classes outside $Z(G)$. Distinct classes K_1, K_2 are adjacent if some $g_1 \in K_1$ and $g_2 \in K_2$ satisfy $g_1 g_2 = g_2 g_1$. The representatives may change from one edge to the next. All graphs here are finite, simple and undirected.

Throughout, S_n acts on $\{1, \ldots, n\}$ and $n \geq 3$. The centre of S_n is trivial, so the vertices are precisely its nonidentity classes. The cycle lengths of a permutation, including a 1 for each fixed point, give a partition λ of n. We write C^λ for its conjugacy class. Two permutations belong to the same class exactly when their cycle lengths agree, with multiplicity. In a partition, i^a denotes a copies of i.

Example 1.1. The four vertices of $\Gamma(S_4)$ are labelled below by their cycle types.



The square of $(1\,2\,3\,4)$ is $(1\,3)(2\,4)$, which in turn commutes with $(1\,3)$. No power of the 4-cycle is a transposition, and only its powers commute with it. A 3-cycle also commutes only with its powers. This gives the path and the isolated vertex shown above.

Britnell and Wildon expressed adjacency in terms of partitions [1, Proposition 4]. We use their criterion to show why a pentagon occurs in degree eight, disappears when a fixed point is added, and reappears in every degree from ten onwards. An induced pentagon needs three colours although it has no triangle. The exceptional graph $\Gamma(S_9)$ has no such obstruction. In fact, each of its induced subgraphs can be coloured with as many colours as its largest clique has vertices.

Herzog, Longobardi and Maj studied the graph on *nonidentity* classes [4]. Saeidi used the same convention to classify the solvable groups whose graph has an isolated vertex, and gave restrictions and examples for nonsolvable groups [8]. Mohammadian and coauthors classified

the finite groups with triangle-free graphs on noncentral classes [5]. Salahshour and Ashrafi treated finite groups in which every noncentral element has abelian centralizer [9]. Cameron and coauthors survey these graphs [2]. For symmetric groups, Ray, Arora and Ma study other forbidden induced subgraphs, including paths on four vertices and claws [7, Theorem 3.1]. In their discussion of S_4 the three nonisolated classes are described as a triangle. The centralizer calculation in Example 1.1 excludes its third edge. The graph in Cameron et al. [2, Figure 1] agrees with the example after removal of the identity vertex.

2 Partitions and commuting classes

A *partition* of n is a non-increasing finite sequence of positive integers with sum n. Let $\mathcal{P}(n)$ denote the set of partitions of n and $p(n) = |\mathcal{P}(n)|$. The vertex labels of $\Gamma(S_n)$ are the partitions other than (1^n). We omit parts of multiplicity zero.

Definition 2.1 (Equal-part coarsening)**.** Let $\lambda, \nu \in \mathcal{P}(n)$. We say that ν is an *equal-part coarsening* of λ, and write

$$\lambda \rightsquigarrow \nu,$$

if ν can be obtained by collecting parts of λ into groups in which all parts have the same size, and replacing every group by the sum of its parts. Singleton groups are allowed, so a part may be left unchanged.

Britnell and Wildon's example [1, §3.1] gives

$$(4, 3^4, 1^2) \rightsquigarrow (12, 4, 2) \quad \text{and} \quad (4, 3^4, 1^2) \rightsquigarrow (6, 4, 3^2, 1^2).$$

In the first coarsening, the four 3's are merged to form 12, the two 1's are merged to form 2, and the 4 is left alone. The rule forbids merging, for example, a 3 with the 1's. A part occurring once survives every equal-part coarsening. A group of parts of size r always produces a multiple of r.

The relation is reflexive but not transitive. Indeed,

$$(6, 4, 3^2, 1^2) \rightsquigarrow (6^2, 4, 2) \quad \text{and} \quad (6^2, 4, 2) \rightsquigarrow (12, 4, 2),$$

but $(6, 4, 3^2, 1^2) \not\rightsquigarrow (12, 4, 2)$. The two parts of size 6 in the intermediate partition arise from parts of different sizes in the original partition and therefore cannot be merged in a single equal-part coarsening. Indeed, no group of equal parts in the original partition can sum to 12.

Theorem 2.2 (Britnell–Wildon [1])**.** *Let $\lambda, \mu \in \mathcal{P}(n)$. The conjugacy classes C^λ and C^μ contain commuting representatives if and only if there is a partition $\nu \in \mathcal{P}(n)$ such that*

$$\lambda \rightsquigarrow \nu \qquad \textit{and} \qquad \mu \rightsquigarrow \nu.$$

Proof. Suppose that permutations x and y of cycle types λ and μ commute. Then $H = \langle x, y \rangle$ is abelian, and its orbits partition $\{1, \ldots, n\}$. On an H-orbit, any element of H that sends one point to another also sends the entire x-cycle through the first point equivariantly onto the x-cycle through the second. Hence all cycles of x on that orbit have the same length. The same holds for y. Consequently, the orbit sizes form a common equal-part coarsening of λ and μ.

Conversely, suppose ν is a common coarsening. Consider one part d of ν. Within the corresponding block, suppose $d = ar = bs$, where r is a part of λ repeated a times and s is a part of μ repeated b times. Identify the block with the cyclic group $\mathbb{Z}/d\mathbb{Z}$. Translation by a has order $d/a = r$, so it has a cycles, each of length r. Similarly, translation by b has b cycles of length s. The two translations commute because all translations of a cyclic group commute. Performing this construction independently on every part of ν gives commuting permutations of cycle types λ and μ. □

In particular, if $\lambda \rightsquigarrow \mu$, then μ is a common coarsening. We use cycle types in place of vertex names from here on. Commuting representatives exist even for two copies of the same type, although a vertex has no loop.

3 Connected components

The *prime graph* $\Pi(G)$ has the prime divisors of $|G|$ as vertices. It joins distinct primes p, q when G contains an element of order pq. Herzog, Longobardi and Maj proved that $\Pi(G)$ and the graph on *nonidentity* conjugacy classes have the same number of components [4, Proposition 9]. This applies to our graph for S_n. The convention matters for a general group. In Q_8, the noncentral classes $\{\pm i\}, \{\pm j\}, \{\pm k\}$ are isolated, and the prime graph has just one vertex.

Lemma 3.1. *Let p and q be distinct primes not exceeding n. Then p and q are adjacent in $\Pi(S_n)$ if and only if*

$$p + q \leq n.$$

Proof. If $p + q \leq n$, a disjoint p-cycle and q-cycle have product of order pq. Conversely, suppose that $g \in S_n$ has order pq. Each cycle length of g divides pq. If one cycle has length pq, then g moves at least $pq \geq p + q$ points. Otherwise, the least common multiple of the cycle lengths can contain both prime factors only if some cycle has length divisible by p and a distinct cycle has length divisible by q. Those two disjoint cycles move at least $p + q$ points. In either case $p + q \leq n$. □

Theorem 3.2. *Let $n \geq 3$. The graph $\Gamma(S_n)$ is disconnected if and only if*

$$n \in \{p, p + 1\}$$

for some prime p. In that case $\Gamma(S_n)$ has exactly two connected components, one of which is the isolated vertex

$$C^{(p,1^{n-p})}.$$

For all other n, the graph is connected.

Proof. For S_3 the transposition and 3-cycle vertices are both isolated. Example 1.1 treats S_4. We may therefore suppose $n \geq 5$.

Suppose first that $n = p$ or $n = p + 1$ for a prime p. By Lemma 3.1, the prime-graph vertex p is isolated in $\Pi(S_n)$. Every other odd prime divisor q of $n!$ satisfies $q \leq p - 2$, and hence $q + 2 \leq n$. Thus every other odd prime is adjacent to the vertex 2, so $\Pi(S_n)$ has exactly two components. The result of Herzog, Longobardi and Maj implies that $\Gamma(S_n)$ also has exactly two components.

It remains to identify an isolated conjugacy class. Let g be a p-cycle, and write $C_{S_n}(g) = \{h \in S_n : hg = gh\}$ for its centralizer. A commuting permutation preserves the support of g, and its value on one point there determines its action on that support. Since $n - p \leq 1$,

$$C_{S_n}(g) = \langle g \rangle.$$

Every non-identity element of $\langle g \rangle$ is again a p-cycle. Hence any element commuting with g lies either in the identity class or in the same conjugacy class as g, so $C^{(p,1^{n-p})}$ is isolated.

Conversely, suppose that neither n nor $n - 1$ is prime. Every odd prime $q \leq n$ then satisfies $q \leq n - 2$, and Lemma 3.1 shows that q is adjacent to 2. Therefore $\Pi(S_n)$ is connected, and so is $\Gamma(S_n)$. □

4 Diameter

For vertices u, v of a connected graph X, let $d_X(u, v)$ be the length of a shortest path joining them. The eccentricity of v is $\mathrm{ecc}_X(v) = \max_{u \in V(X)} d_X(u, v)$, and the diameter of X is the largest eccentricity. In a disconnected graph we take these distances within the specified component. Let $s_j = (j, 1^{n-j})$ be the type of a single j-cycle, let $T = s_2$, and denote the component containing T by $\Gamma^*(S_n)$.

Lemma 4.1. *If λ contains a part $k \geq 2$, then $d(\lambda, s_k) \leq 1$. If also $k \leq n-2$, then $d(\lambda, T) \leq 2$. A permutation commutes with some transposition exactly when its cycle type contains a 2 or at least two 1's.*

Proof. Selecting one k-cycle of a permutation gives a permutation that commutes with it. When $n - k \geq 2$ there is room for a transposition outside the selected cycle. For the last assertion, a permutation commutes with $(i\,j)$ exactly when it preserves the set $\{i, j\}$. It then either fixes both letters or interchanges them. □

Proposition 4.2. *The transposition vertex has eccentricity zero at $n = 3$, two when $n \geq 4$ and $\Gamma(S_n)$ is disconnected, and three when $\Gamma(S_n)$ is connected. In the connected case there is just one vertex at distance three from T.*

Proof. At $n = 3$ the component is a singleton. Take $n \geq 4$. Lemma 4.1 puts every vertex except possibly s_n and s_{n-1} within two steps of T. Write m for the even member of $\{n, n-1\}$. Raising an m-cycle to the power $m/2$ gives the involution of type $B = (2^{m/2}, 1^{n-m})$. Selecting one of its transpositions gives a walk s_m, B, T. Since $m \geq 4$ and $n - m \leq 1$, the last assertion of Lemma 4.1 gives $d(s_m, T) = 2$.

Write L for the odd member of $\{n, n-1\}$. If L is prime, the vertex s_L is isolated by Theorem 3.2. Otherwise L is composite. Let p be its least prime divisor and set $A = (p^{L/p}, 1^{n-L})$. A power of an L-cycle has type A. Selecting one p-cycle and then a transposition on two other letters gives the walk s_L, A, s_p, T.

For the matching lower bound, an L-cycle fixes at most one point and therefore commutes only with its powers. Its nonidentity powers have types $(d^{L/d}, 1^{n-L})$ for $d \mid L$, $d > 1$. Each d is odd. None of these types has a 2 or two fixed points, so no vertex in the closed neighbourhood of s_L can commute with a transposition. Thus $d(s_L, T) = 3$. This is the only remaining vertex. □

Theorem 4.3. *For every $n \geq 8$, the diameter of $\Gamma^*(S_n)$ is four.*

Upper bound. In the disconnected cases Proposition 4.2 gives a path of length at most four through T between any two vertices of the component. Suppose the graph is connected. The same proof works for any pair that omits $E = s_L$, where L is the odd member of $\{n, n-1\}$. Retain p, A, m, B from the preceding proof. We use sequences of commuting types as walks, suppressing consecutive repetitions.

If λ has a part k with $2 \leq k \leq n - p$, use

$$E, \quad A, \quad s_p, \quad s_k, \quad \lambda.$$

A p-cycle and a k-cycle fit on disjoint sets of letters. The last step selects a k-cycle of a representative of λ.

Otherwise every nontrivial part of λ exceeds $n - p$. Since $p \leq \sqrt{L} \leq n/2$, there is just one such part. If λ has at least two fixed points, it commutes with a transposition and $d(E, \lambda) \leq 4$ through T. Apart from E itself, only s_m remains. Set $W = (p^2, 1^{n-2p})$, which selects two p-cycles from A. The types W and B both coarsen to

$$(2p,\, 2^{(m-2p)/2},\, 1^{n-m}).$$

Indeed, W groups its two p's together and pairs the remaining ones. The involution type B groups p twos together. Here $2p \leq m$ since $L \geq p^2$ and $m \geq L - 1$. We obtain the walk E, A, W, B, s_m. □

Lower bound. Put $m = 2\lfloor n/2 \rfloor$ and $\epsilon = n - m$. Write $m = a + b$ with $a, b > 1$ odd, distinct and coprime. To do this, set $h = m/2$ and take $h - 1, h + 1$ when h is even, or $h - 2, h + 2$ when h is odd. In the second case $h \geq 5$. Any common divisor of the two odd numbers divides two or four.

Consider $U = (m, 1^\epsilon)$ and $V = (a, b, 1^\epsilon)$. Both belong to $\Gamma^*(S_n)$ by Proposition 4.2 and Lemma 4.1. We show that their closed neighbourhoods have no common vertex and no edge between them, which excludes paths of lengths one, two and three.

The parts of U are distinct, so its sole coarsening is U. Each vertex in its closed neighbourhood therefore has type $(d^{m/d}, 1^\epsilon)$ with $d > 1$ dividing m. Every further coarsening consists of multiples of d, together with one 1 if $\epsilon = 1$.

The parts of V are also distinct. A type in its closed neighbourhood splits a into equal parts of size $r \mid a$, splits b into equal parts of size $s \mid b$, and retains the optional 1. At least one of r, s exceeds one because the identity is excluded. Interchanging a, b if necessary, assume $r > 1$. Coprimality implies that every occurrence of r comes from the a-block. In any coarsening, the groups formed from these parts produce numbers greater than one whose sum is a. A coarsening shared with a vertex in the closed neighbourhood of U would make every such number divisible by d. This would give $d \mid a$, although $d > 1$ divides $m = a + b$ and $\gcd(a, m) = 1$. By Theorem 2.2, even equality of two vertices across the closed neighbourhoods is impossible. □

At $n = 8$ the lower-bound pair is (8) and $(5, 3)$. One shortest path is

$$(8), \quad (2^4), \quad (2, 1^6), \quad (3, 1^5), \quad (5, 3).$$

For comparison, Woodcock's commuting graph has the individual nonidentity permutations as vertices. When n and $n - 1$ are both composite, that graph has diameter five [11, Theorem 1.1]. Projecting a path to its conjugacy classes gives an upper bound of five here. Theorem 4.3 gives the exact value for the class graph.

5 Clique and chromatic numbers

A *clique* is a set of pairwise adjacent vertices. Let ω_n be the size of a largest clique of $\Gamma(S_n)$ and χ_n the least number of colours needed to colour its vertices with adjacent vertices receiving different colours.

Theorem 5.1. *For $m = \lfloor n/2 \rfloor$ we have*

$$\max\{m, p(m) - 1\} \le \omega_n \le \chi_n \le p(n) - 1.$$

Consequently $\log \omega_n = \Theta(\sqrt{n})$ and $\log \chi_n = \Theta(\sqrt{n})$ as $n \to \infty$.

Proof. The m involution types $(2^i, 1^{n-2i})$, $1 \le i \le m$, all coarsen to $(2^m, 1^{n-2m})$ by pairing fixed points. Hence they form a clique. For the other lower bound, append $n - m$ fixed points to each nonidentity type of S_m. Given two of the resulting $p(m) - 1$ types, realise one on the first m letters and the other on the next m letters, in each case fixing all other letters. These representatives commute.

The vertices in a clique need different colours. Giving each of the $p(n) - 1$ vertices its own colour proves the upper bound. The Hardy–Ramanujan formula gives $\log p(n) \sim \pi\sqrt{2n/3}$ [6]. Applying it at m and at n proves the two growth estimates. □

At $n = 20$ the disjoint-support construction gives $p(10) - 1 = 41$ vertices, compared with the ten involution types. A clique need not admit one coarsening common to all its vertices. At $n = 18$ the largest family sharing a coarsening has 62 vertices, whereas $\omega_{18} = 64$ (Appendix A).

A graph is *weakly perfect* if its clique and chromatic numbers agree. It is *perfect* if they agree for every induced subgraph, obtained by retaining a set of vertices and all edges between them. For example, $\chi_8 = \omega_8 = 7$, although $\Gamma(S_8)$ also contains an induced pentagon, whose clique number is two and chromatic number is three.

Conjecture 5.2. *For every $n \ge 3$, $\chi_n = \omega_n$.*

A clique and a matching colouring certify the conjecture in every degree through 28 (Table 3). These finite certificates leave the question open in general.

6 Perfectness

Theorem 6.1. *For $n \geq 3$, $\Gamma(S_n)$ is perfect if and only if $n \leq 7$ or $n = 9$.*

6.1 An induced pentagon

We first construct the obstruction in degree eight. The edge labels in Figure 1 exhibit commuting representatives by Theorem 2.2.

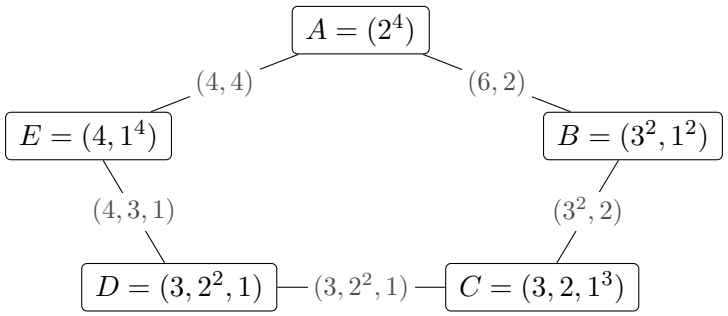


Figure 1: An induced pentagon in $\Gamma(S_8)$. Each edge label is a common equal-part coarsening of its endpoints.

To see why there are no diagonals, observe that C and D each retain a unique 3, whereas A produces only even parts. This excludes AC and AD. The only coarsenings of B are

$$(3^2,1^2), \quad (3^2,2), \quad (6,1^2), \quad (6,2).$$

None is a coarsening of D, which must retain both 3 and 1 and can only keep its two 2's or merge them into 4. None contains the 4 forced by E. This excludes BD and BE. Finally, a common coarsening of C and E would contain $3, 2, 4$, whose sum exceeds eight.

Lemma 6.2 (Adjoining an unused part)**.** *Let $\mathcal{F}$ be a family of nonidentity partitions of $k \geq 3$. Suppose no member contains a part of size t, where $t \geq 1$. Appending a single part t to each member embeds the induced graph on $\mathcal{F}$ as an induced subgraph of $\Gamma(S_{k+t})$.*

Proof. In each extended partition, the new part t occurs once. It must be a singleton group in every coarsening. The other groups give an arbitrary coarsening of the original partition. Thus appending t takes a common coarsening to a common coarsening, and deleting one copy of t from any extended witness gives a common coarsening of the original pair. This cancellation of multisets also works if another group sums to t. □

All five vertices in Figure 1 use only part sizes $1, 2, 3, 4$. Taking $t = n - 8$ therefore gives an induced pentagon for every $n \geq 13$. Appending a fixed point has a different effect. In S_9, the extensions of C and E are $(3, 2, 1^4)$ and $(4, 1^5)$, with common coarsening $(4, 3, 2)$. The additional edge is one reason we must examine degree nine separately.

For $n = 10, 11, 12$ take the rows of Table 1. Each row gives the five types in cyclic order and one witness for each consecutive pair. To exclude the diagonals, C forces a 3 which neither A nor E can produce. The single large part of D (six or eight) cannot occur in a coarsening of A. Its sum with the 5 forced by B exceeds n. Finally E produces only even parts and groups of at most two ones, so it cannot produce the 5 forced by B. This excludes AC, CE, AD, BD, BE, respectively.

n	A	B	C	D	E
10	(5^2)	$(5, 1^5)$	$(3, 1^7)$	$(6, 1^4)$	(2^5)
11	$(5^2, 1)$	$(5, 1^6)$	$(3, 1^8)$	$(8, 1^3)$	$(2^5, 1)$
12	$(5^2, 1^2)$	$(5, 1^7)$	$(3, 1^9)$	$(8, 1^4)$	$(2^5, 1^2)$
n	AB	BC	CD	DE	EA
10	(5^2)	$(5, 3, 1^2)$	$(6, 3, 1)$	$(6, 4)$	(10)
11	$(5^2, 1)$	$(5, 3, 1^3)$	$(8, 3)$	$(8, 2, 1)$	$(10, 1)$
12	$(5^2, 1^2)$	$(5, 3, 1^4)$	$(8, 3, 1)$	$(8, 2, 1^2)$	$(10, 1^2)$

Table 1: Induced pentagons in degrees ten through twelve, with common coarsenings for their edges.

6.2 The positive cases

A vertex is *simplicial* if its neighbours form a clique. Restoring a simplicial vertex to a perfect graph preserves perfectness. Indeed, consider an induced subgraph H containing the restored vertex v. Its neighbours use distinct colours and together with v form a clique, so there are at most $\omega(H) - 1$ such neighbours. A colouring of $H - v$ with at most $\omega(H)$ colours leaves a colour free for v.

We also use that any graph whose vertices split into two cliques is perfect. Its complement is bipartite. A bipartite graph has no induced odd cycle of length at least five and no complement of such a cycle. For length five the complement is another pentagon, and for every larger odd length the complement contains a triangle. The strong perfect graph theorem [3] now applies to both the bipartite graph and its complement.

Proof of Theorem 6.1. For $n \leq 7$ and $n = 9$, Table 2 gives a sequence of simplicial deletions ending in a union of two cliques. Checking these conditions only requires the pairwise adjacency test of Theorem 2.2. Reversing the deletions proves that each of these graphs is perfect. All other degrees contain one of the induced pentagons constructed above. A pentagon has clique number two and chromatic number three, so any graph containing it as an induced subgraph fails to be perfect. □

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Revision and verification

The original undergraduate project contains the perfectness classification, the connectedness criterion and the diameter conjecture. During the November 2026 revision OpenAI Codex developed the proof of the diameter theorem, the exact eccentricity argument, the disjoint-support clique bound and the shorter proof using an unused cycle length. Codex also helped revise the exposition and produced the Python checks and the additional finite certificates. **The Lean formalisation on the paper's web page was generated entirely by AI (OpenAI Codex).** The certificate starts with conjugacy classes of actual permutations and proves that their graph is isomorphic to the partition graph used here. For Theorem 4.3 it proves the upper bound for every pair in the transposition component and exhibits a pair at distance four. It proves perfectness for every induced subgraph in the stated degrees and derives the logarithmic growth estimates from elementary partition bounds. Lean checks every entry of Table 3. The supplement lists the corresponding statements and the commands for checking them.

A Finite certificates and computations

A.1 Perfectness certificates

For each n, list the nonidentity partitions in descending lexicographic order and number them from one. Thus the labels begin with (n) and $(n-1,1)$, and the last label is the transposition type. In Table 2, delete the listed vertices from left to right. At each step the neighbours of the deleted vertex in the remaining graph form a clique. The last two columns partition the survivors into two cliques. Edges between these two cliques are allowed.

n	Simplicial deletions	First clique	Second clique
3	none	1	2
4	1,2,3,4	empty	empty
5	1	2,5	3,4,6
6	1,2,3,4,6	5,7	8,9,10
7	1,2,3,4,5,6,10	7,9,11,13,14	8,12
9	1,2,3,4,5,6,8,9,11,10, 12,13,14,15,16,19	7,20,21,26,27,29	17,18,22,23,24,25,28

Table 2: Certificates for the six perfect graphs in Theorem 6.1.

In degree nine there are 29 vertices and 130 edges. For example, vertex 1 has type (9) and its only neighbour is (3^3). After sixteen deletions, the thirteen remaining vertices form the two cliques in the last row.

A.2 Two graph constructions

The file `computations.sage` contains the original SageMath 10.1 computation [10]. For each partition the routine chooses a number of equal parts to merge, then recurses on the unused parts. It removes duplicate coarsenings and joins two nonidentity partitions when their coarsening sets meet. It computes diameters for $4 \le n \le 20$, clique and chromatic numbers for $3 \le n \le 10$, and perfectness for $3 \le n \le 14$. The same script checks the six positive cases by searching for induced odd cycles of length at least five in each graph and in its complement. It starts at the least vertex of a possible cycle and discards extensions with a chord.

The independent Python construction starts from a possible common coarsening ν. For every part d of ν, choose a divisor $r \mid d$ and replace d with d/r copies of r. All nonidentity partitions obtained from ν in this way form a clique. Taking the union of these edges over $\nu \in \mathcal{P}(n)$ gives exactly $\Gamma(S_n)$. An additional check in degrees three through ten enumerated the centralizer of one representative of each class and recovered the same edges. The README accompanying the source gives commands for the checks.

A.3 Diameters and colouring

Breadth-first searches from each vertex in the transposition component give diameters $0, 2, 3, 3, 3$ for $n = 3, 4, 5, 6, 7$, respectively. They give four throughout $8 \le n \le 20$, in agreement with Theorem 4.3. Figure 2 shows the four nonsingleton components before the diameter reaches four.

For each listed degree the Python checks test every pair in the proposed clique and verify that every edge joins different colours. A matching clique and colouring prove both numbers exact. In degree 18 the largest family sharing any one coarsening has 62 members.

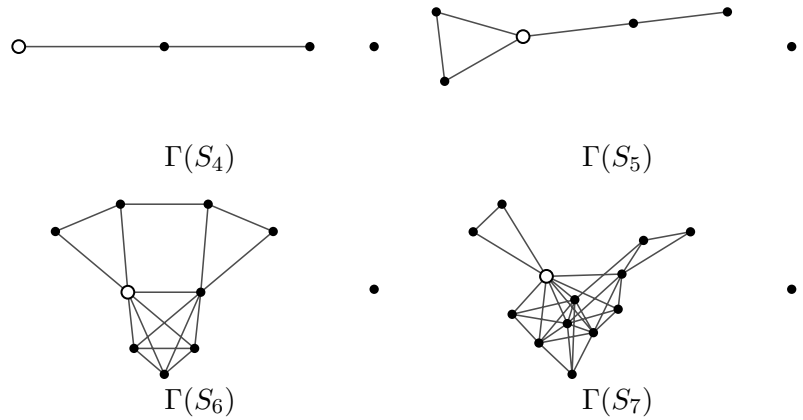


Figure 2: The graphs in degrees four through seven. An open circle marks the transposition class. The isolated vertex on the right of each panel is the long prime-cycle class of Theorem 3.2.

n	3	4	5	6	7	8	9	10	11
$\chi_n = \omega_n$	1	2	3	5	5	7	11	11	15
n	12	13	14	15	16	17	18	19	20
$\chi_n = \omega_n$	22	23	29	45	45	59	64	87	91
n	21	22	23	24	25	26	27	28	
$\chi_n = \omega_n$	111	123	175	175	223	246	295	351	

Table 3: Certified clique and chromatic numbers of $\Gamma(S_n)$.

References

[1] John R. Britnell and Mark Wildon. "Commuting Conjugacy Classes: An Application of Hall's Marriage Theorem to Group Theory". In: *Journal of Group Theory* 12.6 (2009), pp. 795–802. DOI: 10.1515/JGT.2009.013.

[2] Peter J. Cameron et al. "A Survey on Conjugacy Class Graphs of Groups". In: *Expositiones Mathematicae* 42.4, 125585 (2024). DOI: 10.1016/j.exmath.2024.125585.

[3] Maria Chudnovsky et al. "The Strong Perfect Graph Theorem". In: *Annals of Mathematics* 164.1 (2006), pp. 51–229. DOI: 10.4007/annals.2006.164.51.

[4] Marcel Herzog, Patrizia Longobardi, and Mercede Maj. "On a Commuting Graph on Conjugacy Classes of Groups". In: *Communications in Algebra* 37.10 (2009), pp. 3369–3387. DOI: 10.1080/00927870802502779.

[5] Abbas Mohammadian et al. "Triangle-Free Commuting Conjugacy Class Graphs". In: *Journal of Group Theory* 19.6 (2016), pp. 1049–1061. DOI: 10.1515/jgth-2016-0002.

[6] NIST Digital Library of Mathematical Functions. *Unrestricted Partitions*. Equation 27.14.8. URL: https://dlmf.nist.gov/27.14.E8.

[7] Papi Ray, Sonakshee Arora, and Xuanlong Ma. *Forbidden Subgraphs on Conjugacy Class Graphs of Groups*. arXiv:2406.01305v5. Aug. 2, 2026. URL: https://arxiv.org/abs/2406.01305v5.

[8] Amin Saeidi. "Finite Groups Whose Commuting Conjugacy Class Graphs Have Isolated Vertices". In: *Communications in Algebra* 51.2 (2023), pp. 648–656. DOI: 10.1080/00927872.2022.2107213.

[9] Mohammad Ali Salahshour and Ali Reza Ashrafi. "Commuting Conjugacy Class Graph of Finite CA-Groups". In: *Khayyam Journal of Mathematics* 6.1 (2020), pp. 108–118. DOI: 10.22034/kjm.2019.97177.

[10] The Sage Developers. *SageMath, the Sage Mathematics Software System*. Version 10.1. Open-source mathematical software. Aug. 20, 2023. URL: https://github.com/sagemath/sage/releases/tag/10.1.

[11] Timothy Woodcock. "The Commuting Graph of the Symmetric Group S_n". In: *International Journal of Contemporary Mathematical Sciences* 10.6 (2015), pp. 287–309. DOI: 10.12988/ijcms.2015.4553.